# SOME EXACT SOLUTIONS OF THE FERRO-HYDRODYNAMICS EQUATIONS

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The theoretical study of nonisothermal flows of magnetizable liquids presents serious mathematical difficulties, which are intensified as compared to to the study of normal liquids by the necessity of simultaneous solution of both the hydrodynamics and Maxwell's equations, with corresponding boundary conditions for the magnetic field. Thus, in most cases problems of this type are solved by neglecting the effect of the liquid's nonisothermal state on the field distribution within the liquid, and also, as a rule, with use of an approximate solution for Maxwell's equations and fulfillment of the boundary conditions for the field [1-3]. The present study will present easily realizable practical formulations of the problem which permit exact one-dimensional solutions of the complete system of the equations of thermomechanics of electrically nonconductive incompressible Newtonian magnetizable liquids with constant transfer coefficients. A common feature of the formulations is the presence of a longitudinal temperature gradient along the boundaries along which liquid motion is accomplished. Plane-parallel convective flows of this type in a conventional liquid and their stability were considered in [4-6].

1. The basic system of ferrohydrodynamics equations may be written in the following manner (see, e.g., [1]):

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \eta\Delta\mathbf{v} + \rho\mathbf{g} + \mu_0 M\nabla H; \tag{1.1}$$

$$\operatorname{div}\mathbf{v} = 0; \tag{1.2}$$

$$\mathbf{v}\nabla T = \varkappa\Delta T; \tag{1.3}$$

$$\operatorname{rot}\mathbf{H} = 0,\ \operatorname{div}\mathbf{B} = 0,\ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}),\ \mathbf{M} = (M/H)\mathbf{H}; \tag{1.4}$$

$$\rho = \rho^0[1 - \beta(T - T^0)],\ M = M^0 + \chi(H - H^0) - K(T - T^0), \tag{1.5}$$

where $\rho$ is density; **v**, velocity; p, pressure with consideration of magnetostriction terms; $\eta$, viscosity coefficient; **g**, acceleration of gravity; $\mu_0$, magnetic permittivity of a vacuum; **M**, magnetic moment of a unit volume of the liquid; **H**, **B**, magnetic field intensity and induction; T, temperature; $\varkappa$, thermal diffusivity; $\beta$, thermal expansion coefficient; $\chi$, magnetic susceptibility; K, pyromagnetic coefficient with consideration of dependence $M(\rho)$; the index zero denotes some mean value of quantities from which they are evaluated.

The equation for temperature, Eq. (1.3), omits the term describing the magnetocaloric effect, which would produce only an insignificant contribution to the longitudinal temperature gradient in a few of the problems considered below.

Boundary conditions for velocity are derived from the condition of adhesion on the solid boundary and stress balance on the free surface. The temperature distribution on the boundaries is considered specified, while for the magnetic field the tangential component of intensity and the normal component of induction must be continuous on the boundaries.

2. The first problem considers the flow of a liquid in the absence of gravity ($\mathbf{g} = 0$) along an infinite toroidal cylinder (parallel to the z axis), bounded by inner radius $R_1$ and external radius $R_2$, at the boundaries of which there are maintained constant, and generally speaking, differing temperature gradients

$$T = T_1 + \gamma_1 z \quad \text{at} \quad r = R_1,\ T = T_2 + \gamma_2 z \quad \text{at} \quad r = R_2.$$

Such motion is realized in the central portion of a toroidal cylinder with length much greater than its width, with faces bounded by masses maintained at constant but different temperatures.

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Thus, in a cylindrical coordinate system $(r, \varphi, z)$ we consider a one-dimensional, one-directional ($\mathbf{v} = [0, 0, v(r)]$) flow of a magnetizable liquid with temperature distribution $T = T(r, z)$. Maxwell's equations for $r > R_1$ also admit a one-dimensional solution $\mathbf{H} = [0, i/r, 0]$, $\mathbf{M} = [0, M(r, z), 0]$, corresponding to the field created by passage of a constant current $I = 2\pi i$ through an internal cylinder $(r < R_1)$, which exactly satisfies the boundary conditions. The motion of the liquid and the temperature distribution within it do not change this field distribution.

Writing the expression for temperature in the form

$$T = T_1 + (T_2 - T_1)\frac{\ln(r/R_1)}{\ln(R_2/R_1)} + \left[\gamma_1 + (\gamma_2 - \gamma_1)\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\right]z + \Theta(r),$$

we obtain from the original system of equations (1.1), (1.5) for the desired functions $v(r)$, $\Theta(r)$

$$-\frac{\partial p}{\partial r} + \mu_0 M\frac{\partial H}{\partial r} = 0, \quad -\frac{\partial p}{\partial z} + \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0, \quad v\frac{\partial T}{\partial z} = \varkappa\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right) \tag{2.1}$$

with boundary conditions for $\Theta$:

$$\Theta = 0 \quad \text{at} \quad r = R_1,\ R_2.$$

Eliminating the pressure from the first two equations of Eq. (2.1) by cross-differentiation and subtraction, and considering the equation of state (1.5), and the explicit form of the field and temperature, we obtain for $v(r)$ an ordinary third-order differential equation with a known free term:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)\right] = \frac{\mu_0 Ki}{\eta r^2}\left[\gamma_1 + (\gamma_2 - \gamma_1)\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\right],$$

the general solution of which is

$$v = -\frac{\mu_0 Ki}{\eta}\left[\left(\gamma_1 - \frac{\gamma_2 - \gamma_1}{\ln(R_2/R_1)}\right)r + \frac{\gamma_2 - \gamma_1}{\ln(R_2/R_1)}r\ln\frac{r}{R_1}\right] + c_1 r^2 + c_2\ln\frac{r}{R_1} + c_3. \tag{2.2}$$

The three arbitrary constants $c_1$, $c_2$, $c_3$ are determined from the two boundary conditions at $r = R_1$, $R_2$, and by specifying the liquid flow rate through the channel cross section. If the channel is open, the flow motion may be brought about by an external pressure difference between the ends. In the case of a closed channel where forced convection is absent, and motion is produced solely by the thermomagnetic mechanism of free convection, no liquid should be expended and the third constant is found from the condition

$$\int_{R_1}^{R_2} v\,dr = 0. \tag{2.3}$$

Most striking in Eq. (2.2) is the fact that the presence of a constant radial pressure difference $T_2 - T_1$ has no effect on the velocity distribution of the convective motion.

For a specified velocity profile Eq. (2.2), integration in general form of the third equation of Eq. (2.1) for temperature offers no difficulty in principle, although it is cumbersome:

$$\Theta = \frac{1}{\varkappa}\int_{R_1}^{r}\frac{1}{r}\left\{\int rv\left[\gamma_1 + (\gamma_2 - \gamma_1)\frac{\ln(r/R_1)}{\ln(R_2/R_1)}\right]dr\right\}dr + c_4\ln\frac{r}{R_1}.$$

The boundary condition $\Theta = 0$ at $r = R_1$ is fulfilled automatically, and the constant $c_4$ is determined by the second boundary condition.

On the basis of the solutions obtained, we will consider some concrete situations. First of all let the temperature gradients on both channel boundaries be identical ($\gamma_1 = \gamma_2 = \gamma$) and both boundaries be rigid ($v = 0$ at $r = R_1$, $R_2$). Then, introducing the dimensionless coordinate $x = (r - R_1)/(R_2 - R_1)$ with range from 0 to 1 and dedimensionalizing the velocity with the scale factor $\varkappa/(R_2 - R_1)$, we obtain the distribution of the dimensionless velocity $V$ of the closed convective flow in the channel in the form

$$V = -\frac{\mathrm{Ra}_m}{\delta(2+\delta)}\{x - x^2 + c[x(2+\delta x)\ln(1+\delta) - (2+\delta)\ln(1+\delta x)]\},$$
$$c = \delta/2[2(\delta^2 + 3\delta + 3)\ln(1+\delta) - 3\delta(2+\delta)], \tag{2.4}$$

where $\delta = (R_2 - R_1)/R_1$; $\mathrm{Ra}_m = \mu_0 Ki\gamma(R_2 - R_1)^4/\eta\varkappa R_1^2$ is the magnetic Rayleigh number.

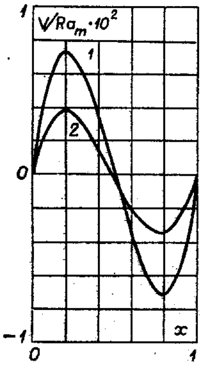


Fig. 1

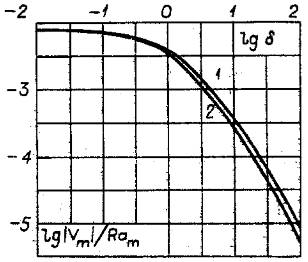


Fig. 2

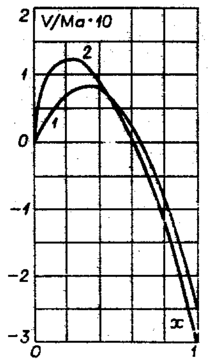


Fig. 3

For a small channel width ($\delta \ll 1$) solution (2.4) transforms to the well-known solution describing plane-parallel free convective flow in a horizontal layer with longitudinal temperature gradient under the influence of gravity [4]:

$$V = (Ra_m/12)(2x^3 - 3x^2 + x),$$

in which the role of the conventional Rayleigh number is played by $Ra_m$ and the motion actually occurs in a magnetic field with constant gradient $i/R_1^2$ equivalent to the acceleration of gravity g.

The velocity profiles calculated from Eq. (2.4) for various values of $\delta$ are presented in Fig. 1 (curve 1 corresponds to $\delta = 0.1$; curve 2, $\delta = 1$). For the same values of $Ra_m$ with increase in $\delta$ the intensity of the motion decreases. The dependence of maximum velocity $V_m$ in the left half of the channel $x < 0.5$ (curve 1) and in the right half (curve 2) is shown in Fig. 2.

The following situation, which is of special interest, develops when the external boundary of the liquid ($r = R_2$) is free, while the internal boundary, as before, is rigid. We stress that a cylindrical free surface is completely natural for a magnetizable liquid in the given case, i.e., in the absence of gravity and in a magnetic field with a radial intensity gradient. Then the boundary condition for velocity on the free surface, with consideration of the temperature dependence of the surface tension coefficient ($\alpha = \alpha^0 - \sigma(T - T^0)$), which can be obtained from the equation of stress balance on the boundary, will have the form [7]

$$\eta \left(\frac{\partial v}{\partial r}\right)_{r=R_2} = \frac{\partial \alpha}{\partial T}\frac{\partial T}{\partial z} = -\sigma\gamma \; (\gamma_1 = \gamma_2 = \gamma).$$

The surface tension gradient on the free boundary produces a thermocapillary convection mechanism [7], which acts together with the magnetic forces.

The solution (2.2) for the given boundary conditions appears as follows:

$$v = -\frac{\mu_0 K i\gamma}{\eta}(r - R_1) + \left(\frac{\mu_0 K i \gamma}{2\eta R_2} - \frac{\sigma\gamma}{2\eta R_2}\right)(r^2 - R_1^2) + D\left(\ln\frac{r}{R_1} - \frac{r^2 - R_1^2}{2R_2^2}\right).$$

The constant D is determined from the condition of closed flow (2.3).

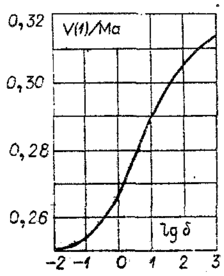


Fig. 4

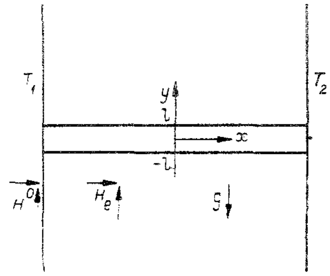


Fig. 5

From the equation obtained it follows, first of all, that the effect of the thermocapillary mechanism is insignificant at $\sigma < \mu_0\mathrm{Ki}$, independent of channel geometry. At the same time, this effect is dominant at $\sigma > \mu_0\mathrm{Ki}(R_2/R_1)$. The dimensionless velocity distribution in the channel for purely thermocapillary convection is given by the expression

$$V = -\frac{\mathrm{Ma}}{2(1+\delta)}\{x(2+\delta x) + D[2(1+\delta^2)\ln(1+\delta x) - \delta x(2+\delta x)]\}, \tag{2.5}$$
$$D = -\delta(3+\delta)/[6(1+\delta)^3\ln(1+\delta) - 6\delta(1+\delta)^2 - \delta^2(3+\delta)],$$

where $\mathrm{Ma} = \sigma\gamma(R_2 - R_1)^2/\eta\varkappa$ is the Marangoni number.

This velocity profile is depicted in Fig. 3 (curve 1 corresponds to $\delta = 0.1$, curve 2 to $\delta = 100$). With increase in $\delta$ at the same value of Ma the intensity of the motion on the free surface increases. The dependence of V(1) on $\delta$ is shown in Fig. 4.

At small channel width ($\delta \ll 1$) Eq. (2.5) transforms to the known expression for plane-parallel flow [4]:

$$V = (\mathrm{Ma}/4)(2x - 3x^2).$$

3. We will consider the flow of a magnetizable liquid in a horizontal plane-parallel channel cut off by a massive ferromagnetic having pyromagnetic coefficient K and magnetic susceptibility $\chi$ the same as those of the liquid. These last requirements are significant in obtaining exact one-dimensional solutions of the Maxwell equations and satisfying the boundary conditions. The geometry of the problem is depicted in Fig. 5. A temperature difference is maintained across the walls of the mass, ensuring a constant longitudinal gradient $\gamma$ in the boundaries of the layer: $T = T^0 + \gamma x$ at $y = \pm l$. The entire system is located in an external homogeneous magnetic field $H^0$, normal or tangential to the lateral boundaries of the mass. Then in the central portion of the mass, according to Maxwell's equations (1.4), (1.5) in the first case the following distribution of field $H_e$ and magnetization $M_e$ are realized (values outside the liquid layer will be denoted by a subscript e):

$$H_{ey} = M_{ey} = 0, \quad H_{ex} = H_e^0 + [K\gamma/(1+\chi)]\,x,$$
$$H_e^0 + M_e^0 = H^0, \quad M_{ex} = M_e^0 - [K\gamma/(1+\chi)]\,x \tag{3.1}$$

while in the second case

$$H_{ex} = M_{ex} = 0, \quad H_{ey} = H_e^0 = H^0, \quad M_{ey} = M_e^0 - K\gamma x. \tag{3.2}$$

In the central part of the channel we will have a plane-parallel convective liquid motion $v_y = 0$, $v_x = v(y)$ and temperature distribution $T = T^0 + \gamma x + \vartheta(y)$, while in the boundary layers $x = \pm l$, $v = \vartheta = 0$.

In accordance with Maxwell's equations and the boundary conditions, the field and magnetization distribution within the liquid will be described by the following expressions:

a) for external magnetic field $H_e$ parallel to the layer (Eq. (3.1))

$$H_y = M_y = 0, \quad H_x = H_i^0 + [K\gamma/(1+\chi)]\,x, \tag{3.3}$$
$$H_i^0 = H_e^0, \quad M_x = M^0 - [K\gamma/(1+\chi)]\,x - K\vartheta(y);$$

b) for external magnetic field $H_e$ perpendicular to the layer (Eq. (3.2))

$$H_x = M_x = 0, \quad H_y = H_i^0 + [K/(1+\chi)]\,\vartheta(y), \tag{3.4}$$
$$H_i^0 + M^0 = H_e^0 + M_e^0, \quad M_y = M^0 - K\gamma x - [K/(1+\chi)]\,\vartheta(y).$$

We note that all inhomogeneities in the field and magnetization are produced by the temperature inhomogeneity. In the first case the magnetic field intensity has only a horizontal gradient, in the second, only a vertical gradient.

To find the desired functions v(y) and $\vartheta$(y) the system of equations (1.1)-(1.5) gives

$$v\gamma = \varkappa\vartheta''; \tag{3.5}$$

a) $\mathbf{H} = [H(x),\ 0,\ 0]$

$$-\partial p/\partial x + \eta v'' + \mu_0 M\partial H/\partial x = 0,\ -\partial p/\partial y - \rho g = 0; \tag{3.6}$$

b) $\mathbf{H} = [0,\ H(y),\ 0]$

$$-\partial p/\partial x + \eta v'' = 0,\ -\partial p/\partial y - \rho g + \mu_0 M\partial H/\partial y = 0, \tag{3.7}$$

where the primes denote differentiation with respect to y.

Eliminating the pressure from Eqs. (3.6), (3.7) and introducing the dimensionless coordinate $\xi = y/l$, velocity $V = (l/\varkappa)v$, and temperature $\Theta = \vartheta/\gamma l$, and considering Eqs. (3.3), (3.4) we obtain

$$V = \Theta'',\ \text{a)}\ V''' - \text{Ra}_m\Theta' - \text{Ra} = 0,\quad \text{b)}\ V''' + \text{Ra}_m\Theta' - \text{Ra} = 0 \tag{3.8}$$

and boundary conditions $V = \Theta = 0$ at $\xi = \pm 1$.

The conventional Ra and magnetic $\text{Ra}_m$ Rayleigh numbers are defined in the present case in the following manner: $\text{Ra} = \beta g\gamma l^4\rho/\varkappa\eta$, $\text{Ra}_m = \mu_0 K^2\gamma^2 l^4/\eta\varkappa(1+\chi)$. We note, first, that the value of $\text{Ra}_m$ is always positive, and second, that the force producing convection in the given situation is gravitation. For Ra = 0 the problem of Eq. (3.8) becomes an eigenvalue problem and gives the value of the critical $\text{Ra}_m$, corresponding to disruption of mechanical equilibrium, equal to $(\pi/2)^4$. This is true only of case a) where the magnetic field is parallel to the layer. In case b) mechanical equilibrium of the liquid is stable. For Ra $\neq$ 0 motion develops at any value, no matter how small.

System (3.8) is equivalent to the equations describing plane-parallel convective flow of a conventional liquid in an inclined layer with a longitudinal temperature gradient [6]: a) with an acute angle between the temperature gradient and force of gravity; b) when the angle is obtuse. This calls to mind the fact that in both cases the effect of the magnetic field is equivalent to a longitudinal component of the gravity, despite the fact that in the first case the field has only a longitudinal, or in the second case, transverse, intensity component within the liquid. From the example considered it is clear that the action of a magnetic field intensity gradient on thermal convection of a magnetizable liquid is not always equivalent to a gravitational force acting in the same direction, in cases where the gradient is produced by a temperature inhomogeneity.

The solutions of Eq. (3.8) with specified boundary conditions corresponding to closed flow have the form

$$V = -\frac{1}{2}\frac{\text{Ra}}{\sqrt{\text{Ra}_m}}\left(\frac{\sin\varepsilon\xi}{\sin\varepsilon} - \frac{\text{sh}\,\varepsilon\xi}{\text{sh}\,\varepsilon}\right),\ \varepsilon = \sqrt[4]{\text{Ra}_m},$$

$$\Theta = \frac{\text{Ra}}{\text{Ra}_m}\left[\frac{1}{2}\left(\frac{\sin\varepsilon\xi}{\sin\varepsilon} + \frac{\text{sh}\,\varepsilon\xi}{\text{sh}\,\varepsilon}\right) - \xi\right];$$

$$V = \frac{\text{Ra}}{\sqrt{\text{Ra}_m}S}\left(\frac{\sin\varepsilon\xi\,\text{ch}\,\varepsilon\xi}{\sin\varepsilon\,\text{ch}\,\varepsilon} - \frac{\cos\varepsilon\xi\,\text{sh}\,\varepsilon\xi}{\cos\varepsilon\,\text{sh}\,\varepsilon}\right),$$

$$\Theta = -\frac{\text{Ra}}{\text{Ra}_m}\left[\frac{1}{S}\left(\frac{\sin\varepsilon\xi\,\text{ch}\,\varepsilon\xi}{\cos\varepsilon\,\text{sh}\,\varepsilon} + \frac{\cos\varepsilon\xi\,\text{sh}\,\varepsilon\xi}{\sin\varepsilon\,\text{ch}\,\varepsilon}\right) - \xi\right],$$

$$S = \text{tg}\,\varepsilon\,\text{cth}\,\varepsilon + \text{ctg}\,\varepsilon\,\text{th}\,\varepsilon,\quad \varepsilon = \sqrt[4]{\text{Ra}_m/4}$$

and, as was noted earlier, have been studied a number of times for conventional liquids (see, e.g., [6]).

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# EFFECT OF SPECIMEN TEMPERATURE ON THE BREAKING POINT FOR SPLITTING-OFF IN AMG-6 ALUMINUM ALLOY

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The investigation of the temperature dependence of the strength of constructional materials under intense shock loads, including breaking loads, is of considerable practical interest. There are certain technical difficulties involved in loading the materials and making the required measurements in such experiments. Hence, the number of papers devoted to this problem is extremely limited. In [1, 2] data is given on the elastic–plastic properties of a number of metals at normal and high temperatures, obtained by investigating the parameters of elastic waves excited by an explosion. In [3, 4] investigations were carried out of the temperature dependence of the breaking point for splitting-off in steel and copper. In the present paper we investigate the effect of the specimen temperature on the breaking point for splitting off in widely used AMG-6 aluminum alloy in the temperature range from 0°C to 550°C, i.e., practically up to the temperature at which the alloy begins to melt.*

The specimens investigated were cut from a single blank and were disks 70 mm in diameter and 10 mm thick with a conical side surface (at an angle of 45°).

The specimens were tested on special equipment, a diagram of which is shown in Fig. 1.

Specimen 1 was heated by the radiant heat flux from a Nichrome filament heater 2 with a power of 3 kW (50 A, 60 V), mounted on a heat-resistant screen. The temperature of the specimen was monitored with a thermocouple 3 up to the instant when the specimen was loaded. The time taken to heat the specimen up to a temperature of 550°C was ~20 min. The nonuniformity of the temperature over the specimen thickness at the instant of loading did not exceed ~5°C. The heated specimen was displaced by means of a cable 4 along the direction 5 on a special platform 6 under the loading device 7. The specimen was loaded by a shock aluminum plate (110 × 150 × 4 mm), scattered up to the required velocity by a glancing detonation wave from a layer of explosive placed on it, initiated simultaneously along one of the faces of the plate from the explosive. To prevent splitting-off in the striker the latter was separated from the explosive by a layer of porous material. The velocity of the striker was varied by varying the thickness of the layer of explosive, and simultaneity of the shock on the surface of the specimen was achieved by placing the striker at a certain angle to the specimen depending on the velocity of the striker. The split-off plates formed as a result of the loading were collected in practically undamaged form using a porous damper 8 of low rigidity placed in a steel container 9.

The method of determining the breaking point was as follows. A shock load was applied to the specimen and the presence or absence of split-off was observed visually after the experiment. (If necessary the specimen was cut along an axis, a thin section was made, and metallographic analysis was carried out.) By a gradual

*The melting of alloys and solid solutions is characterized by a melting temperature range. For AMG-6 alloy the temperature at which melting begins is ~570°, and the temperature at which melting ends is ~640°C [5].

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